Ch4: Spectral theory

→ Definition 1

A is $n \times n$ matrix and $X \in \mathbb{R}$ be vector nonzero.

$AX = \lambda X$ $\lambda \in \mathbb{R}$: eigen value of A
$X \in \mathbb{R}$: eigen vector of A

→ to find eigen values (λ): $\det(\lambda I - A) = 0$ identity matrix

→ Finding eigen vectors and eigen values

1. Find the eigen value λ by solving the characteristic equation:
   $\det(\lambda I - A) = 0$

2. For each λ, find the basic eigen vector $X \neq 0$ by finding the basic solution to: $(\lambda I - A) X = 0$

3. To verify your work make sure that $AX = \lambda X$ for each λ.

Ex1: $A = \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix}$ Find its eigenvalues and eigenvectors

1. Characteristic eqt:

$\det(xI - A) = 0$

$$\left| x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} x+5 & -2 \\ 7 & x-4 \end{pmatrix} \right| = 0$$

$\Rightarrow (x+5)(x-4) - (7\times(-2)) = 0$

$x^2 - 4x + 5x - 20 + 14 = 0$

$x^2 + x - 6 = 0$

$\Rightarrow$ $x_1 = -3$ (λ_1)

$x_2 = 2$ (λ_2)

2. For $\lambda_1 = -3$

$(\lambda_1 I - A)X = 0$

$$\left[-3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ 7 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -2 & 0 \\ 7 & -7 & 0 \end{array}\right)$$

$R_1/2 \to R_1$ $$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 7 & -7 & 0 \end{array}\right)$$

$R_2 - 7R_1 \to R_2$ $$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$y = t$

$x - t = 0 \Rightarrow x = t$

$$X \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$

$(\lambda_2 I - A) X = 0$

$$\left[\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 7 & -2 & 0 \\ 7 & -2 & 0 \end{array}\right)$$

$R_1/7 \to R_1$ $$\left(\begin{array}{cc|c} 1 & -2/7 & 0 \\ 7 & -2 & 0 \end{array}\right)$$

$R_2 - 7R_1 \to R_2$ $$\left(\begin{array}{cc|c} 1 & -2/7 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$y = t$

$x - \frac{2}{7}t = 0$

$x = \frac{2}{7}t$

$$X \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{7}t \\ t \end{pmatrix} = t \begin{pmatrix} \frac{2}{7} \\ 1 \end{pmatrix}$$

Ex2 : $A = \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}$ Find the eigenvalues and eigenvectors of A

1. $\det(xI - A) = 0$

$$\left| \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix} \right|$$

$$\left| \begin{pmatrix} x-5 & 10 & 5 \\ -2 & x-14 & -2 \\ 4 & 8 & x-6 \end{pmatrix} \right|$$

$$(x-5)\begin{vmatrix} x-14 & -2 \\ 8 & x-6 \end{vmatrix} - 10 \begin{vmatrix} -2 & -2 \\ 4 & x-6 \end{vmatrix} + 5 \begin{vmatrix} -2 & x-14 \\ 4 & 8 \end{vmatrix}$$

$$(x-5)\left[(x-14)(x-6)+16\right] - 10\left[-2(x-6)+8\right] + 5\left(-16-(4(x-14))\right) = 0$$

$$(x-5)(x^2-6x-14x+84) - 10(-2x+12+8) + 5(-16-4x+56) = 0$$

$$(x-5)(x^2-20x+84) - 10(-2x+20) + 5(-4x+40) = 0$$

$$(x-5)(x^2-20x+100) + (20x-\cancel{200}) + (-20x+\cancel{200}) = 0$$

$$(x-5)(x-10)^2 = 0$$

$x_1 = 5 \; (\lambda_1)$

$x_2 = 10 \; (\lambda_2)$

For $\lambda_1 = 5$

$(\lambda_1 I - A) X = 0$

$$\left[\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 10 & 5 & 0 \\ -2 & -9 & -2 & 0 \\ 4 & 8 & -1 & 0 \end{array}\right) \quad R_1 \leftrightarrow R_2 \quad \begin{pmatrix} -2 & -9 & -2 & 0 \\ 0 & 10 & 5 & 0 \\ 4 & 8 & -1 & 0 \end{pmatrix}$$

$$R_1/-2 \quad \begin{pmatrix} 1 & 9/2 & 1 & 0 \\ 0 & 10 & 5 & 0 \\ 4 & 8 & -1 & 0 \end{pmatrix} \quad R_3 - 4R_1 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & 9/2 & 1 & 0 \\ 0 & 10 & 5 & 0 \\ 0 & -10 & -5 & 0 \end{array}\right)$$

$$R_3 + R_2 \to R_3 \quad \left(\begin{array}{ccc|c} 1 & 9/2 & 1 & 0 \\ 0 & 10 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_2/10 \quad \left(\begin{array}{ccc|c} 1 & 9/2 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = t$

$y + \frac{1}{2}t = 0 \Rightarrow y = -\frac{1}{2}t$

$x + 9/2\left(-\frac{1}{2}t\right) + t = 0$

$x - \frac{5}{4}t = 0 \Rightarrow x = \frac{5}{4}t$

$$X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} \frac{5}{4} \\ -1/2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 10$

$(\lambda_2 I - A)X = 0$

$$\left[\begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 5 & 10 & 5 & 0 \\ -2 & -4 & -2 & 0 \\ 4 & 8 & 4 & 6 \end{array}\right) \quad \xrightarrow{R_1/5} \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & -4 & -2 & 0 \\ 4 & 8 & 4 & 0 \end{array}\right)$$

$R_2 + 2R_1 \to R_2$

$R_3 - 4R_1 \to R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = t$

$y = s$

$x + 2s + t = 0$

$x = -2s - t$

$$X = \begin{pmatrix} -2s - t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -2s \\ 0 \\ t \end{pmatrix} + \begin{pmatrix} -t \\ s \\ 0 \end{pmatrix}$$

→ Definition 2

- A and B are similar matrices if there exist P invertible matrix such that $P^{-1}BP = A$

- $\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$

ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$\text{trace}(A) = 1 + 5 + 9$

- if A and B are similar:
  → $\det(A) = \det(B)$
  → $\text{trace}(A) = \text{trace}(B)$

→ Diagonalisable

A $n \times n$ matrix is diagonalisable if there exist an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix;

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

إذا P invertible طلع يعني
أن A diagonalisable

Theorem:

A $n \times n$ matrix is diagonalisable if and only if there exist invertible matrix P given by:

$P = [x_1 \ \ x_2 \ \dots \ x_n]$

where x are eigen vector of A.

Ex3 : $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{pmatrix}$ a) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$

$\det(xI - A) = 0$

$$\left| \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} x-2 & 0 & 0 \\ -1 & x-4 & 1 \\ 2 & 4 & x-4 \end{vmatrix} = 0$$

$(x-2)\begin{vmatrix} x-4 & 1 \\ 4 & x-4 \end{vmatrix} \Rightarrow (x-2)\left[(x-4)(x-4) - 4\right] = 0$

$(x-2)(x^2 - 4x - 4x + 16 - 4) = 0$

$(x-2)(x^2 - 8x + 12) = 0$

$x^3 - 8x^2 + 12x - 2x^2 + 16x - 24 = 0$

$x^3 - 10x^2 + 28x - 24 = 0$

$\Rightarrow x_1 = 2 \quad (\lambda_1)$

$x_2 = 6 \quad (\lambda_2)$

For $\lambda_1 = 2$

$(\lambda_1 I - A)X = 0$

$$\left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{pmatrix} \right] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 2 & 4 & -2 & 0 \end{array}\right) \quad R_1 \leftrightarrow R_3 \quad \left(\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\frac{R_1}{2} \to R_1 \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_2 + R_1 \to R_2 \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = s$

$y = t$

$x + 2t - s = 0$

$x = -2t + s$

$$X = \begin{matrix} x \\ y \\ z \end{matrix} = \begin{pmatrix} -2t + s \\ t \\ s \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$$

$$= t \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{X_1} + s \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{X_2}$$

For $\lambda_2 = 6$

$(\lambda_2 I - A) X = 0$

$$\left[\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{pmatrix}\right] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \end{array}\right) \quad R_2 + \frac{1}{4} R_1 \to R_2 \quad \left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \end{array}\right)$$

$$R_3 - \frac{R_1}{2} \to R_3 \quad \left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{array}\right) \quad R_3 - 2R_2 \to R_3 \quad \left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\frac{R_2}{2} \to R_2 \quad \left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \frac{R_1}{4} \to R_1 \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z = t$

$y = -\frac{1}{2}t$

$x = 0$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2}t \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix}_{X_3}$$

$$P\,[X_1, X_2\, X_3] = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix}$$

$$P^{-1} = \left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

P is invertible

A is diagonalisable

$$P^{-1}AP = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Theorem :

- $P^{-1}AP = D \Rightarrow A = PDP^{-1}$
- $A^n = [PDP^{-1}]^n$

Ex : $A^2 = [PDP^{-1}]^2 = \underbrace{PDP^{-1} \times P}_{I}DP^{-1}$

$= PD^2P^{-1}$

$\Rightarrow A^n = PD^nP^{-1}$

Ex3: b) Find A^{50}

$$A^{50} = PD^{50}P^{-1}$$

$$A^{50} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2^{50} & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 6^{50} \end{pmatrix} \begin{pmatrix} -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & -1/4 \end{pmatrix}$$